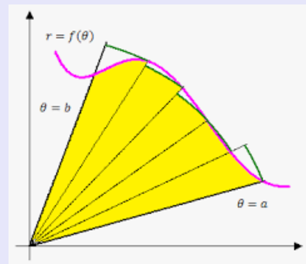
Calculus II

Lecture 23



Feb 19-8:47 AM

Class Qz 21

Use the $\boxed{\text{integral test}}$ to determine whether the series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ is convergent or divergent.

✓ $f(x) > 0$

Let $f(x) = \frac{e^{\frac{1}{x}}}{x^2}$ $x \geq 1$ ✓ $f(x)$ is cont. on $[1, \infty)$

$f'(x) = \frac{e^{\frac{1}{x}} \cdot \frac{-1}{\cancel{x^2}} \cdot \cancel{x^2} - e^{\frac{1}{x}} \cdot 2x}{(x^2)^2}$ ✓ $f'(x) < 0 \rightarrow f(x)$ is decreasing

$= \frac{-e^{1/x} - e^{1/x} \cdot 2x}{x^4} = \frac{-e^{1/x}(1+2x)}{x^4} < 0$

$\int_1^\infty f(x)dx = \lim_{t \to \infty} \int_1^t \frac{e^{\frac{1}{x}}}{x^2} dx = \cdots = \boxed{e-1}$

$u = \frac{1}{x}$ $du = \frac{-1}{x^2} dx$

$\int \frac{e^{1/x}}{x^2} dx = \int_1^{\frac{1}{t}} -e^u du$

Converges

$\therefore \sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ converges by integral test.

Jul 22-7:10 AM

Class QZ 20 $\boxed{\text{Open Notes}}$

Find $\boxed{\text{exact Ans}}$ for surface area when the length of the curve given by $\boxed{x=3t^4,\ y=4t^3}$ for $0\le t\le\sqrt{3}$ is rotated about x-axis.

$$S=\int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

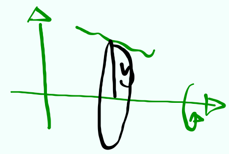
$\frac{dx}{dt}=12t^3$

$\frac{dy}{dt}=12t^2$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=144t^6+144t^4=144t^4(t^2+1)$$

$$S=\int_0^{\sqrt{3}}2\pi\cdot 4t^3\cdot\sqrt{144t^4(t^2+1)}\,dt=\int_0^{\sqrt{3}}2\pi\cdot 4t^3\cdot 12t^2\sqrt{t^2+1}\,dt$$

$$=96\pi\int_0^{\sqrt{3}}t^5\sqrt{t^2+1}\,dt \Rightarrow$$



Jul 18-10:33 AM

$$=96\pi\int_0^{\sqrt{3}}t^5\sqrt{t^2+1}\,dt$$

$u=t^2+1$ $\quad du=2t\,dt$

$t^2=u-1$ $\quad \frac{du}{2}=t\,dt$

$\boxed{u=\sqrt{t^2+1}\ ,\ \text{trig. Subs.}}$

$t=0\to u=0^2+1=1$

$t=\sqrt{3}\to u=(\sqrt{3})^2+1=4$

$$=96\pi\int_0^{\sqrt{3}}t^4\sqrt{t^2+1}\cdot t\,dt=96\pi\int_1^4(u-1)^2\sqrt{u}\,\frac{du}{2}$$

$$=48\pi\int_1^4(u^2-2u+1)\sqrt{u}\,du=48\pi\left[\frac{u^{7/2}}{7/2}-\frac{2u^{5/2}}{5/2}+\frac{u^{3/2}}{3/2}\right]\Big|_1^4$$

$$=48\pi\left[\frac{2}{7}u^3\sqrt{u}-\frac{4}{5}u^2\sqrt{u}+\frac{2}{3}u\sqrt{u}\right]\Big|_1^4$$

$$=48\pi\left[\left(\frac{2\cdot 4^3\cdot\sqrt{4}}{7}-\frac{4\cdot 4^2\sqrt{4}}{5}+\frac{2\cdot 4\sqrt{4}}{3}\right)-\left(\frac{2}{7}-\frac{4}{5}+\frac{2}{3}\right)\right]=$$

$$=48\pi\left[\frac{256}{7}-\frac{128}{5}+\frac{16}{3}-\frac{2}{7}+\frac{4}{5}-\frac{2}{3}\right]=$$

$$=48\pi\left[\frac{254}{7}-\frac{124}{5}+\frac{14}{3}\right]=48\pi\cdot\frac{1696}{105}=\boxed{\frac{27136\pi}{35}}$$

Jul 18-10:33 AM

More on Power Series

Consider

$$\sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + x^3 + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots$$

Infinite geometric Series

First term $a = 1$

Common Ratio $r = x$

Converges is $|r| < 1$

$|x| < 1$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1-x}$$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad -1 < x < 1}$$

Jul 22-8:25 AM

Write $\frac{1}{1+x^3}$ as power Series

Recall $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n \qquad |-x^3| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n (x^3)^n \qquad |x^3| < 1$$

$\vdots$

$-1 < x < 1$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^{3n}}$$

Jul 22-8:29 AM

Write $\frac{x^2}{2-x}$ as a power series.

Some Algebra

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{x^2}{2-x} = x^2 \cdot \frac{1}{2-x}$$

$$= x^2 \cdot \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{x^2}{2} \cdot \frac{1}{1-\frac{x}{2}}$$

$$= \frac{x^2}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

Converges $\left|\frac{x}{2}\right| < 1$

$-2 < x < 2$

$$= \sum_{n=0}^{\infty} \frac{x^2}{2} \cdot \frac{x^n}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+2}}{2^{n+1}}$$

Jul 22-8:33 AM

Write $f(x) = \frac{x}{9+x^2}$ as a power series.

$$\frac{x}{9+x^2} = \frac{x \cdot 1}{9\left(1+\frac{x^2}{9}\right)} = \frac{x}{9} \cdot \frac{1}{1-\left(\frac{-x^2}{9}\right)}$$

$$= \frac{x}{9} \sum_{n=0}^{\infty} \left(\frac{-x^2}{9}\right)^n$$

Converges if

$\left|-\frac{x^2}{9}\right| < 1$

$\left|\frac{x^2}{9}\right| < 1$

$\boxed{-3 < x < 3}$

$$= \sum_{n=0}^{\infty} \frac{x}{9} \cdot \frac{(-1)^n \cdot x^{2n}}{9^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{9^{n+1}}$$

Jul 22-8:39 AM

Write $f(x)=\frac{1}{x^2+4x+3}$ as power Series.

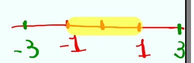
use partial fractions $\qquad f(x)=\frac{-\frac{1}{2}}{x+3}+\frac{\frac{1}{2}}{x+1}$

$$\frac{1}{x^2+4x+3}=\frac{A}{x+3}+\frac{B}{x+1} \qquad =\frac{1}{2}\left[\frac{1}{x+1}-\frac{1}{x+3}\right]$$

$$1=A(x+1)+B(x+3)$$

$x=-1 \to 1=2B \to B=\frac{1}{2}$

$x=-3 \to 1=-2A \to A=\frac{-1}{2}$



$$\frac{1}{x+1}=\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n \qquad -1<x<1$$

$$\frac{1}{x+3}=\frac{1}{3+x}=\frac{1}{3(1+\frac{x}{3})}=\frac{1}{3}\cdot\frac{1}{1-(\frac{-x}{3})}=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n$$

$-3<x<3$

$$f(x)=\frac{1}{x^2+4x+3}=\frac{1}{2}\sum_{n=0}^{\infty}\left[(-x)^n+\frac{(-x)^n}{3^{n+1}}\right] \qquad -1<x<1$$

Jul 22-8:46 AM

write $\ln(1+x)$ in power Series.

$$\ln(1+x)=\int\frac{1}{1+x}\,dx$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n = 1+x+x^2+x^3\dots$$

$$=\int\frac{1}{1-(-x)}\,dx$$

$$=\int\left[1-x+x^2-x^3+x^4\dots\right]dx$$

$$\ln(1+x)=\overset{\frac{x^1}{1}}{\boxed{x}}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots+C$$

if $x=0 \qquad \cancelto{0}{\ln 1}=0-0+0-0+\dots+C$

$C=0$

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Jul 22-8:57 AM

Write $\ln(4-x)$ as power Series. Find Interval of Convergence.

$$\ln(4-x) = \int \frac{-1}{4-x}\,dx$$

$u = 4-x$

$du = -dx$

Domain

$4-x>0$

$x<4$

$$= -\int \frac{1}{4-x}\,dx$$

$$= -\int \frac{1}{4\left(1-\frac{x}{4}\right)}\,dx = \frac{-1}{4}\int \frac{1}{1-\frac{x}{4}}\,dx$$

$$= -\frac{1}{4}\int \left[1 + \frac{x}{4} + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{4}\right)^3 + \cdots\right]dx$$

$$= \frac{-1}{4}\left[\frac{x^1}{1} + \frac{x^2}{4\cdot 2} + \frac{x^3}{4^2\cdot 3} + \frac{x^4}{4^3\cdot 4} + \cdots\right] + C$$

$$= \frac{-1}{4}\sum_{n=1}^{\infty} \frac{x^n}{4^{n-1}n} = -\sum_{n=1}^{\infty} \frac{x^n}{4^n\cdot n} + \boxed{\ln 4}$$

$$= -\sum_{n=1}^{\infty} \left(\frac{x}{4}\right)^n \cdot \frac{1}{n} + \ln 4$$

Ratio Test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{\cancel{x^{n+1}}}{\cancel{4^{n+1}}\cdot(n+1)}}{\frac{\cancel{x^n}}{\cancel{4^n}\cdot n}}\right| = \left|\frac{x\cdot n}{4\cdot(n+1)}\right| = \left|\frac{x}{4}\right|\cdot\frac{n}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x}{4}\right|\left(\frac{n}{n+1}\right)^{\to 1} = \left|\frac{x}{4}\right|\cdot 1 = \left|\frac{x}{4}\right| < 1$$

$$-1 < \frac{x}{4} < 1$$

$$-4 < x < 4$$

Jul 22-9:03 AM

Write $f(x) = \tan^{-1}x$ as power Series.

$$\tan^{-1}x = \int \frac{1}{1+x^2}\,dx$$

$$\frac{1}{1-x}$$

$-1<x<1$

$$= \int \frac{1}{1-(-x^2)}\,dx$$

$$= \int \left[1 - x^2 + x^4 - x^6 + x^8 \cdots\right]dx$$

$$\tan^{-1}x = \frac{x^1}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \cdots + C$$

$x=0$ $\quad \tan^{-1}0 = 0 - 0 + 0 - 0 + 0 \cdots + C$

$\boxed{C=0}$

$$\tan^{-1}x = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^{2n-1}}{2n-1}$$

Jul 22-9:18 AM

$f(x) = \boxed{x} \tan^{-1} x^2$, write as a power Series

$u = x^2 \quad du = 2x\,dx$

$\tan^{-1} x^2 = \int \frac{2x}{1+(x^2)^2} dx$

$\int \frac{1}{1+u^2} du = \tan^{-1} u + C$
$= \tan^{-1} x^2$

$\frac{2x}{1+x^4} = 2x \cdot \frac{1}{1-(-x^4)} = 2x \cdot \sum (-x^4)^n$

$\int \frac{2x}{1+x^4} dx = \int 2x\left[1 - x^4 + x^8 - x^{12} + x^{16} \cdots\right] dx$

$= 2\left[\frac{x^2}{2} - \frac{x^6}{6} + \frac{x^{10}}{10} - \frac{x^{14}}{14} + \cdots\right] + C$

$= 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n-2}}{4n-2} + C$

$\tan^{-1} x^2 = \cancel{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n-2}}{\cancel{2}(2n-1)} + C$

$(x^2)^{2n-1} = x^{2(2n-1)} = x^{4n-2}$

From last example $\tan^{-1} x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$

Replace x with x^2 $\quad \tan^{-1} x^2 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x^2)^{2n-1}}{2n-1}$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n-2}}{2n-1}$

Jul 22-9:27 AM

Write $f(x) = \ln \frac{1+x}{1-x}$ as power Series.

$f(x) = \ln(1+x) - \ln(1-x)$

$= \int \frac{1}{1+x} dx + \int \frac{-1}{1-x} dx$

$= 1 - \cancel{x} + x^2 - \cancel{x^3} + x^4 \cdots + 1 \cancel{+x} + x^2 \cancel{+x^3} + x^4 \cdots$

$= \int (2 + 2x^2 + 2x^4 + 2x^6 + \cdots) dx$

$= 2\int (1 + x^2 + x^4 + x^6 + \cdots) dx$

$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + C\right.$

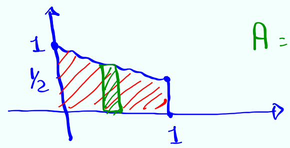
$x = 0$
$C = 0$

$= 2 \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$

Jul 22-9:42 AM

Find the area below $f(x)=\frac{1}{1+x^5}$,

above x-axis from $x=0$ to $x=1$.



$$A=\int_0^1 \frac{1}{1+x^5}\,dx$$

I.C.
$-1<x<1$

$$\frac{1}{1+x^5}=\frac{1}{1-(-x^5)}=1-x^5+x^{10}-x^{15}\cdots$$

$$\int \frac{1}{1+x^5}\,dx=\int 1-x^5+x^{10}-x^{15}\cdots\,dx$$

$$=x-\frac{x^6}{6}+\frac{x^{11}}{11}-\frac{x^{16}}{16}+\cdots\Big|_0^1$$

Estimate within 2 decimal places

$$1-\frac{1}{6}+\frac{1}{11}-\frac{1}{16}+\cdots$$

$$\approx\boxed{.86}$$

Jul 22-9:52 AM